Research Statement: Transitions of Geometries and Groups

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0.1 Abstract

My research concerns the study of transitions between different *homogeneous spaces,* G/H, associated with a fixed Lie group, G, obtained by taking limits of conjugates of the subgroup H. The idea of geometric transition may be studied from the perspectives of geometry, topology, algebraic geometry, and dynamics. For example:
- How does one geometry transition to another at infinity?
- What are the possible cusps at infinity on a manifold?
- In the Chabauty topology on the space of all subgroups of a group, under what conditions does one group limit to another?
- What properties characterize limits of the diagonal subgroup of the general linear group?

I also study Dessin d'Enfant, graphs which describe branched coverings of Riemann surfaces. If the covering is Galois, the ramification type is *regular.* We characterize when small changes to regular ramification types are realizable.

0.2 Intuition and Motivation

Imagine blowing up a ball with air so that eventually the ball is so large, it looks like the earth. Locally, the ball looks flat. This example is given in [6]: a sequence of spheres tangent to a plane, with increasing radius, will limit to the tangent plane in the Hausdorff topology on closed sets. Such a process is an example of a *geometric transition*, or a continuous path of geometric structures that changes type in the limit.

There are several ways of making the idea of inflating a ball mathematically precise. Envision the curvature of the ball approaching zero. Or, define a way to measure coordinates on the ball, and then use coordinates to describe the radius of the ball increasing. A sphere is *intrinsically* different from the plane. On a sphere, the angles in a triangle will sum up to more than 180 degrees, since the edges bulge outwards. In the plane, the angles in a triangle sum to exactly 180 degrees. This property about triangles is *intrinsic* to the geometry of the space, and will hold true no matter how large or small the triangle is. Blowing up a ball is an example of a transition between two different kinds of geometry: spherical and Euclidean.

The idea of continuously deforming one kind of geometry into another appears in many areas of mathematics and physics, see [6]. Homogeneous spaces have groups of isometries which are Lie groups, and have an associated Lie algebra, with multiplication given by the bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. This bilinear map is determined by the action on a basis, and hence by *structure constants*. The structure constants may continuously change, as long as $[,]$ still determines a Lie algebra. This is related to the theory of Inönü-Wigner contractions in physics, see [4]. Physicists use deformations of Lie algebras in several ways, for example the "classical limit" in relativity where the speed of light, $c \to \infty$; and in transitioning from quantum mechanics to Newtonian mechanics, when $\hbar \to 0$.

Thurston conjectured and Perelman proved: every compact 3-dimensional manifold is composed of pieces, each of which has one of 8 kinds of 3-dimensional geometry, two of which are spherical and Euclidean, [30]. These *Thurston geometries* are (almost) subgeometries of real projective geometry and one may study geometric transitions in this context as paths of conjugacies, [6]. I study geometric transitions given by *conjugacy limits*.

Definition 1. *Let G be a Lie group. A subgroup, $H \leq G$, limits under conjugacy to another subgroup, $L \leq G$, if there is a sequence of conjugating matrices, (P_n) in G, such that $P_n H P_n^{-1} \to L$ in the Hausdorff topology.*

0.3 Limits of the Cartan Subgroup in $SL_3(\mathbb{R})$, and use of the Hyperreals

In my thesis, and published in [22, 24], I studied geometric transitions of the diagonal Cartan subgroup in $SL_n(\mathbb{R})$, (the group of diagonal matrices), building on work of Haettel, [16]. For example, when $n = 3$, a diagonal matrix with distinct eigenvalues determines a projective triangle, since each eigenvector of the matrix designates a vertex of the triangle. The geometric transitions of the diagonal group are determined by identifying the vertices or edges of the triangle to obtain a *degenerate* triangle, [22].

Theorem 2 (Leitner [22])**.** *1. Any subgroup of $SL_3(\mathbb{R})$ isomorphic to $\mathbb{R}^2$ is conjugate to exactly one of the following groups:*

$$\overset{C}{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}}, \quad \overset{F}{\begin{pmatrix} a & t & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}}, \quad \overset{N_1}{\begin{pmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{N_2}{\begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{N_3}{\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}}$$

where $a, b \in \mathbb{R}_{>0}$ and $s, t \in \mathbb{R}$.

2. *Each of these groups is a conjugacy limit of the Cartan subgroup.*

3. *The set of conjugacy classes of limit groups is in bijection with the set of equivalence classes of degenerate triangles in figure 1.*

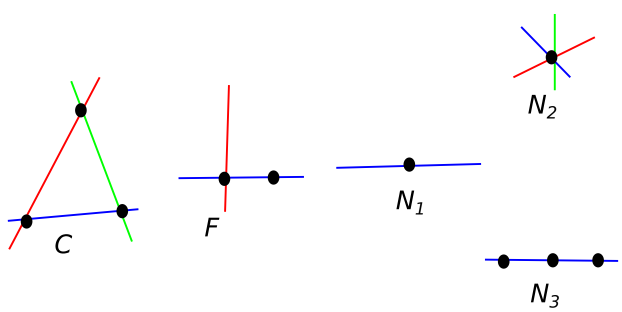

Figure 1: The 5 equivalence classes of degenerate triangles in $\mathbb{R}P^2$

Open Problem 3. *Classify conjugacy limits of the Cartan subgroup in $SL_n(\mathbb{R})$.*

I have done this for $n \leq 4$. There are two directions to approach the general case. One idea is to induct on dimension, by exploring the different ways of adding a vertex infinitesimally close to the (hyperreal) configuration of $n-1$ points in a dimension lower. Another approach is to begin with n points in general position, and to look at the different ways in which these configurations can degenerate.

To study degenerate triangles, I use the *hyperreals*, a non-Archimedean ordered field containing the reals, which includes numbers that are infinitesimally small, and others that are infinitely large. The hyperreals provide a convenient method of imagining and describing phenomena that appear after an *infinite* amount of time, by giving a precise way to measure the infinities involved. Constructing the hyperreals requires the axiom of choice [13]. The hyperreals are formed by taking equivalence classes of sequences of real numbers, much in the way that the reals are formed by taking Cauchy sequences of rational numbers. However, in this case, the equivalence relation is given by taking a non-principal ultra-filter. The motivation for working over the hyperreals is to eliminate the need to use sequences and take limits going to infinity. The way certain groups transition can be described by *triangles* with infinitesimal sides and angles, and such geometric transitions are controlled by very precise measurements of these quantities.

A sequence of matrices determines a single hyperreal matrix, which represents a hyperreal projective transformation taking the standard basis (triangle for $n = 3$), to a nonstandard basis (infinitesimal triangle for $n = 3$). Instead of taking the limit of images of the diagonal group under conjugation by a sequence of matrices, consider conjugating the diagonal group by a single hyperreal matrix, and take the shadow: this is like projecting back down into the real numbers, and eliminating the infinitesimal information. Using this approach, I classified nonstandard triangles.

Theorem 4 (Leitner, [22]). *A limit group of the Cartan subgroup in $SL_3(\mathbb{R})$ is controlled by two hyperreal ratios: the length of the longest side of the nonstandard triangle, and the ratio of the largest infinitesimal angle to the largest side.*

A nonstandard triangle fits in one of five equivalence classes, corresponding to the 5 limit groups, see Figure 1. As a result, I classified the conjugacy limits of the diagonal Cartan subgroup under *any* sequence of matrices, see [22]. Using work of [16, 19] I showed

Theorem 5 (Leitner [23]). *There are precisely 15 subgroups of $SL_4(\mathbb{R})$ isomorphic to $\mathbb{R}^3$, up to conjugacy. Each is a limit of the Cartan subgroup.*

Future research: Apply hyperreal techniques to classify limits of the Cartan subgroup in $SL_n(\mathbb{R})$.

0.4 Limits of the Cartan in $SL(n, \mathbb{Q}_p)$

In joint work in progress with Corina Ciobotaru [5], we extended some of these results to $SL(n, \mathbb{Q}_p)$. Instead of groups acting on projective space, the groups act on the Bruhat-Tits building for $SL(n, \mathbb{Q}_p)$.

Theorem 6 (Ciobotaru- Leitner, [5]). • *For $p \neq 2$ there are precisely two limits of diagonal Cartan subgroup in $SL(2, \mathbb{Q}_p)$ up to conjugacy.*
• *For $p \neq 3$ there are 5 limits of the diagonal Cartan in $SL(3, \mathbb{Q}_p)$ up to conjugacy.*
• *For $p \neq 2$ there are 17 limits of the diagonal Cartan in $SL(4, \mathbb{Q}_p)$ up to conjugacy.*

The restriction on p is from Hensel's lemma.

Our results are as follows: The diagonal group is the center of a stabilizer of an apartment in the building. If a group contains hyperbolic elements, then a flat torus theorem implies it stabilizes a flat in the Bruhat-Tits building. If a group contains no hyperbolic elements, then it is contained in the unipotent radical of a parabolic group that stabilizes a facet in the spherical building at infinity. The question of which limit groups can limit to others is answered by the combinatorial containments of the facets in the spherical building at infinity.

Future work: Extend the classification for $n \geq 5$ and generalize to other local fields.

0.5 The Topology on the Space of Limit Groups: $Red(n) \subset Ab(n)$

The set of all closed subgroups of a group is a topological space with the *Chabauty topology* on closed sets: [9, 17, 16]. Define two topological subspaces with the subspace topology, $\widehat{Ab}(n)$: the set of all subgroups of $SL_n(\mathbb{R})$ isomorphic to $\mathbb{R}^{n-1}$; and following notation in [19], let $\widehat{Red}(n)$ be the space of all limits of the Cartan subgroup in $SL_n(\mathbb{R})$. Quotient by conjugacy in $SL_n(\mathbb{R})$ to obtain two spaces with the quotient topology: $Ab(n)$ and $Red(n)$. Since limit groups of the Cartan subgroup are isomorphic to $\mathbb{R}^{n-1}$, then $Red(n) \subset Ab(n)$.

Suprenko and Tyshkevitch, [29], classified conjugacy classes of maximal commutative nilpotent subalgebras over $\mathbb{C}$, for $n \leq 6$. They showed $Ab(5)$ is finite, and so $Red(5)$ is finite. Iliev and Manivel, [19], ask if $Red(n)$ is finite when $n \geq 6$. I found an invariant that shows:

Theorem 7 (Leitner [24])**.** *If $n \geq 7$, then $0 < dim(Red(n)) = O(n^2)$.*

Thus for $n \geq 7$ there are infinitely many non-conjugate limits of the Cartan subgroup. When $n \leq 5$ there are finitely many conjugacy classes of limits of the Cartan subgroup. The case $n = 6$ is open.

When $n \leq 4$, by Theorems 2 and 5 $Ab(n) = Red(n)$. In [24] I found the first explicit examples of elements of $Ab(n) - Red(n)$ for $n = 5, 6$, which may be extended for $n \geq 7$. Previously, Haettel [16] and Iliev and Manivel [19] gave a dimension counting argument which shows $Red(n) \subsetneq Ab(n)$ for $n \geq 7$.

Theorem 8 (Leitner [24])**.** *$Ab(n) = Red(n)$ if and only if $n \leq 4$. When $n \geq 5$, then $Red(n) \subsetneq Ab(n)$.*

In [5] we extend Theorem 7 to $\mathbb{Q}_p$. The analog of Theorem 8 is computationally more difficult since we cannot apply results of [29] and need to compute for $n = 5, 6$.

Future research: Over any local field, when is an abelian group a limit group? What special properties do limit groups have? I hope to find an invariant of abelian groups that distinguishes limit groups.

I plan to explore the the topology of the spaces $Ab(n)$ and $Red(n)$. For example: How many components do $Ab(n)$ and $Red(n)$ have? Does every component of $Ab(n)$ contain a component of $Red(n)$? Is $Red(n)$ a retract of $Ab(n)$?

0.6 Varieties of Closed Subgroups

Often it is useful to distinguish conjugacy classes of abelian Lie subgroups of $GL_n(\mathbb{R})$. A matrix, $A \in GL_n(\mathbb{C})$, has a Jordan Normal Form (JNF) that uniquely determines the conjugacy class of the matrix. In general, one cannot simultaneously put *two* matrices into JNF in the same basis. But for matrices contained in the same abelian subgroup, one can. Currently, I am studying a Jordan Normal Form Invariant for an *abelian Lie subgroup* G of $GL_n(\mathbb{R})$. The invariant is a function on the projective space $\mathbb{P}(G)$ given by projectivising the group G. Level sets of this function are projective semi-algebraic varieties. The goal is to obtain a practical method to distinguish the conjugacy class of G. Varieties connected to the Jordan Normal Form invariant have long been studied as *rank varieties* and *varieties of commuting matrices*, see [3, 11, 12, 28].

The subspace of conjugates of the diagonal group has closure which is a (semi-algebraic) variety $\mathcal{V}$, called the *Chabauty compactification* of the associated homogeneous space. These methods give information about the dynamics of the action of $GL_n(\mathbb{R})$ on $\mathcal{V}$.

For $n = 3$, Haettel showed $\mathcal{V}$ is a CW complex with 2-skeleton the wedge sum of $\mathbb{R}P^2$ and S^2, see [16]. The cells of the CW complex correspond to conjugacy classes of groups, with dimension equal to the dimension of the Borel group minus the dimension of the normalizer of the subgroup. Going to infinity in the limit and changing to another group lowers the dimension of the cell. The attaching maps of the cell complex correspond to the possible limits of a group.

Future research: Determine the Chabauty compactification in higher dimensions over different fields.

In general, one might ask if the Chabauty compactification of G/H and the subalgebra compactifiation $\mathfrak{g}/\mathfrak{h}$ are homeomorphic for different closed subgroups $H \subset G$. Guivarc'h- Ji-Taylor [15] showed that these are homeomorphic when $H = K$ the maximal compact subgroup. Haettel [16] has shown the same for the diagonal subgroup in $SL(n, \mathbb{R})$. A natural question is when these compactifications are homeomorphic for different fields and subgroups. In [5] we give strong evidence that these compactifications are the same for the Cartan subgroup in $SL(n, \mathbb{Q}_p)$.

0.7 Properties of Chabauty Limits

Even more generally, one might ask which properties of a group are preserved under taking a conjugacy limit. Let G be a Lie group (not necessarily $SL_n(\mathbb{R})$). Suppose $H \leq G$ is any closed subgroup and there is a sequence $(g_n) \subset G$ such that $g_n H g_n^{-1} \to L$. For example, if H is abelian, is L abelian? What about if H is nilpotent, or reductive? Or properties of groups which are not algebraic, such as distal or amenable?

0.8 Generalized Cusps on Convex Projective Manifolds

Another application of these ideas is to study *generalized cusps* on convex projective manifolds, see [1, 7, 8]. A *convex projective manifold* $C = \Omega/\Gamma$ is the quotient of convex subset of projective space, Ω, by a discrete group of projective transformations $\Gamma \subset PGL(n+1, \mathbb{R})$. A *generalized cusp* in dimension 3 is a convex projective manifold that is the product of a ray and a torus. The holonomy centralizes a 1-parameter subgroup of $PGL_n(\mathbb{R})$. Using the classification of limits of the Cartan subgroup, one may attempt to classify all generalized cusps together with *transitions* between different types of generalized cusp.

Theorem 9 (Leitner, [23]). *A generalized cusp on a properly convex projective 3-dimensional manifold is projectively equivalent to one of 4 possible cusps.*

Generalized cusps on projective manifolds also give rise to affine structures on the torus, see [14, 25]. For a generalized cusp $C = \Omega/\Gamma$ in dimension n, we require that ∂C is compact and strictly convex (contains no line segment) and that there is a diffeomorphism $h : [0, \infty) \times \partial C \to C$.

Together with Sam Ballas and Daryl Cooper in [2] we classified generalized cusps in dimension n, and explored new geometries arising from such cusps.

Theorem 10 (Ballas-Cooper-Leitner [2]). *The holonomy of a generalized cusp is a lattice in one of a family of Lie groups $G(\lambda)$ parameterized by a point $\psi = (\psi_1, ..., \psi_n) \in \mathbb{R}^n$, with $\psi_1 \geq \cdots \geq \psi_n \geq 0$.*

A generalized cusp may be determined either by its group of isometries, or by a properly convex projective domain. More generally a maximal-rank cusp in a hyperbolic n-orbifold is determined by the similarity class of lattice in $Isom(\mathbb{E}^{n-1})$. We parametrize the space of lattices, and use this to describe transitions between cusps. Let $\mathcal{M}od^n$ denote the collection conjugacy classes of unmarked lattices in holonomy groups of generalized cusps.

Theorem 11 (Ballas-Cooper-Leitner [2]). *There is a bijection between elements of $\mathcal{M}od^n$ and projective classes of generalized cusps.*

We show every generalized cusp is foliated by $(n-1)$-dimensional manifolds with a Euclidean structure, and every generalized cusp deformation retracts to a hyperbolic cusp. We also discuss the volume of cusps with respect to the Hausdorff measure induced by the Hilbert metric.

Theorem 12 (Ballas-Cooper-Leitner [2]). *A generalized cusp has finite volume if and only if there exists $k \leq n-2$ such that $\psi_i = 0$ for all $i \geq k$.*

Future research: One might ask which structures are possible on a given manifold. For example, it is shown in [1, 21] that there are properly convex projective structures on the Figure-8 knot complement with three of the types of generalized cusp in dimension 3. At the time of writing it is not known if it also admits one of the fourth type. We want to determine which cusp types can be realized on convex projective manifolds. In future work we also intend to show the map in Theorem 11 varies continuously.

0.9 Almost-Regular Dessin d'Enfant

A branched covering $\tilde{\Sigma} \to \Sigma$ of closed connected surfaces is locally modeled on $z \mapsto z^d$, and d is the degree of the map. This is a covering map away from the finitely many branch points. If the ith branch point has m_i preimages, write the local degrees as $[d_1, ..., d_{m_i}]$ a partition of d. For example, the data for the covering $P^1\mathbb{C} \to P^1\mathbb{C}$ given by $x \mapsto x^3$ as $[3], [1,1,1], [3]$. Each branched covering determines a branch datum.

A branch datum is *compatible* if it satisfies the Riemannn-Hurwitz formula restricting the genus, and some orientability conditions. The Riemann-Hurwitz formula tells us how the genus of Σ and $\tilde{\Sigma}$ match, with a correction term for the ramified points e_p. Here χ denotes the Euler characteristic.

$$\chi(\tilde{\Sigma}) = d \cdot \chi(\Sigma) - \sum_{p \in \Sigma} (1 - e_p).$$

If a covering $\tilde{\Sigma} \to \Sigma$ exists, then the branch datum is compatible. The *Hurwitz existence problem* is to determine which compatible branch data are realized by branched coverings.

The Riemman Existence Theorem says that a branch covering exists if and only if the corresponding *Dessin* can be drawn. Given a Dessin, we can construct a branch datum. Together with Joachim König and Danny Neftin in [20], we found exceptional branch datum by proving certain Dessin cannot be drawn.

A Dessin d'Enfant is a connected graph where each vertex is assigned one of two colors and the endpoints of any edge are different colors. Moreover, there is a cyclic ordering of the edges around any vertex. See [27, 31]. As a result of Corollary 4.3 in [26], there are 4 finite types of regular ramifications (the ramification degree at each point is equal), these are branched coverings from the torus to the sphere.

A $[2^{\frac{n}{2}}][4^{\frac{n}{4}}][4^{\frac{n}{4}}]$ **B** $[2^{\frac{n}{2}}][3^{\frac{n}{3}}][6^{\frac{n}{6}}]$ **C** $[2^{\frac{n}{2}}][2^{\frac{n}{2}}][2^{\frac{n}{2}}][2^{\frac{n}{2}}]$ **D** $[3^{\frac{n}{3}}][3^{\frac{n}{3}}][3^{\frac{n}{3}}]$.

Here n is the degree of the map, and the exponent is the number of entries of that type. The Dessin for these ramification types correspond to regular tilings of a torus by hexagons and squares. We study when it is possible to change the branch data slightly to an *almost-regular ramification type* where the ramification degree over each point is the same except for a bounded number of entries, called the error, ε.

Theorem 13 (König-Leitner-Neftin [20]). *Almost regular ramification types in genus 0 and 1 with $\varepsilon \leq 6$ for $A - D$ and $\varepsilon \leq 10$ for C-D are realizable, except 4 nonrealizable types in genus 1, and 1 nonrealizable type in genus 0.*

We showed that the genus 1 types are realizable as *local changes* to a tiling of the torus by regular hexagons or squares, where we make changes to only finitely many tiles. Answering this question for larger ε using the techniques we developed would be a good starting point for an undergraduate research project.

0.10 Broader Applications

Answers to the Hurwitz problem are useful in solving problems which reduce to the case of determining the existence of a map between Riemann surfaces with certain ramification type, or the existence of branched coverings, or rational functions of certain ramification type.

Geometric transitions may be used to understand the interplay between different types of geometry, by describing the range of possible geometries within some given parameters. For instance, geometric transitions give a sense in which some geometries are more nilpotent than others. Furthermore, diagonal subgroups and matrices are central to much of mathematics. A classification of such subgroups might be useful to many different areas of research.

In order to discuss the geometry of a manifold as a whole, one can consider it modeled as a subgeometry of projective space. Classifying the types of geometry of the ends of convex projective structures is progress towards proving a higher dimensional geometrization theorem.

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